# STAT 339: HOMEWORK 2 (PROBABILITY BASICS)

DUE ON BLACKBOARD BY START OF CLASS ON FRIDAY. FEB. 24

**Instructions.** Turn in your writeup and code to Blackboard as an archive file (e.g., `.zip`, `.tar`, `.gz`) by the start of class on Friday 2/24.

As always, you may use any language you like for the progamming components of this assignment — the tasks are stated in a language-neutral way. You may also use any typesetting software to prepare your writeup, but the final document should be a PDF. LaTeXis encouraged; a reproducible research format in which code is embedded into the document (e.g., knitr, RMarkdown, Jupyter or IPython Notebook) is even more encouraged.

1. (Warmup) A discrete random variable, $Y$, has a distribution given by the following table.

| $y$ | $P(y)$ |
|---|---|
| 0 | 0.2 |
| 1 | 0.4 |
| 2 | 0.2 |
| 3 | 0.1 |
| 4 | 0.1 |

   (a) Calculate $P(1 < Y \leq 3)$.

   (b) Calculate $\mathbb{E}[Y]$.

   (c) Calculate $\mathbb{V}\text{ar}[Y]$.

   (d) Let $W = 2Y + 3$. Calculate $\mathbb{E}[W]$.

   (e) Calculate $\mathbb{V}\text{ar}[W]$.

*Date*: Last Revised: February 18, 2017.

2. (Warmup) Let X and Y be jointly distributed discrete random variables. Their joint probability distribution is given in the following table:

| $x$ | $y$ | | | | | $P(x)$ |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | |
| 1 | 0.02 | 0.04 | 0.06 | 0.08 | 0.05 | |
| 2 | 0.08 | 0.02 | 0.10 | 0.02 | 0.03 | |
| 3 | 0.05 | 0.05 | 0.03 | 0.02 | 0.10 | |
| 4 | 0.10 | 0.04 | 0.05 | 0.03 | 0.03 | |
| $P(y)$ | | | | | | |

   (a) Find the marginal distribution of $X$.

   (b) Find the marginal distribution of $Y$.

   (c) Calculate the conditional probability $P(X = 3|Y = 1)$.

3. **Poisson distribution.** A taqueria is trying to model the sales of their fish tacos. Assume that the number of customers that buy fish tacos in a given hour can be modeled by a Poisson distribution with parameter $\lambda$, which has a distribution given by

$$P(k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

   (where $k!$ is $k$-factorial: $k! := k \times (k-1) \times \cdots \times 2 \times 1$) Suppose $\lambda = 2.7$. What is the probability that, in on hour,

   (a) exactly one customer buys fish tacos?

   (b) more than three customers buy fish tacos?

   (c) between one and four (inclusive) customers buy fish tacos?

4. **Game Show.** A game show allows contestants to win money by spinning a wheel. The wheel is divided into four equally sized spaces, three of which contain dollar amounts and one of which says BUST. The contestant can spin as many times as they want, adding any money won to their total, but if they land on BUST, the game ends and they lose everything won so far.

   (a) Suppose for the moment that the contestant keeps going until they go bust. Let $W$ be the random variable that represents the number of successful spins before going bust. Indicate the valid range for $W$, and find a formula that gives $P(W = w)$ for any $w$ (be sure to specify the domain where your formula applies).

(b) The dollar amounts on the non-BUST spaces are \$100, \$500, and \$5000. Let $X_n$ be the random variable denoting the amount of money won or lost on the $n^{\text{th}}$ spin. Find $\mathbb{E}[X_1]$ (note that there is nothing to lose on the first spin).

(c) Let $S_n$ be the cumulative amount won after $n$ spins. Find $\mathbb{E}[X_{n+1}|S_n]$, as a function of $S_n$. How large does $S_n$ have to get before the expected gain by spinning again is negative?

5. **Triangular density.** A "triangular density" on the interval $[-1, 1]$ is given by the piecewise linear expression

$$p(x) = \begin{cases} k(1-x), & x \geq 0 \\ k(1+x), & x < 0 \end{cases}$$

which yields an isosceles triangle with a peak at 0.

(a) Find the value of $k$ that makes this a valid density (Note: you do not necessarily need to do any integration, although you can — a simple geometric argument is sufficient)

(b) Find $P(|X| > \frac{1}{2})$ (again, a geometric argument is enough).

6. **Semicircular density.** Consider a circle with radius $r$ and center $(0, 0)$ on a plane. The points on the circle satisfy, $x^2 + y^2 = r^2$ (where $x$ is the horizontal coordinate and $y$ is the vertical one). If we take just the "upper half" of the circle, so that $y > 0$, we can solve the circle equation for $y$ in terms of $x$ and $r$ to get $w = \sqrt{r^2 - y^2}$. Treating $r$ as a constant, we can write $g(x)$ in place of $y$ to yield the function $g(x) = \sqrt{r^2 - x^2}$.

(a) What is the valid domain on which $g(x)$ is defined (as a real number)?

(b) Suppose $X$ is a random variable that takes values in the domain defined in the previous part. For any fixed $r$, there is a unique constant, $k_r$, such that $p(x) = k_r \cdot g(x)$ is a valid probability density function. Find $k_r$. (Again, you can work this out with geometry — no calculus is necessary.)

7. **Mean of a Beta Distribution** (Adapted from FCMA Ex. 3.5) If a random variable $R$ has a beta density

$$p(r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1},$$

derive an expression for the expected value of $r$, $\mathbb{E}_{p(r)}\{r\}$. You will need the following identity for the gamma function:

$$\Gamma(x+1) = x\Gamma(x).$$

(The gamma function generalizes the factorial function to all real numbers except non-positive integers, where if $n$ is a positive integer, then $\Gamma(n) = (n-1)!$)

Hint: use the fact that

$$\int_0^1 r^{\alpha-1}(1-r)^{b-1}\ \mathrm{d}r = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

8. **Coding Probability Computations.** Suppose we have a random variable, $Y$ that has a $\mathcal{P}\text{ois}(\lambda)$ distribution.
   (a) Write a function that takes $k$ and $\lambda$ as inputs and returns $P(Y = k)$.
   (b) Write a function that takes $k$ and $\lambda$ as inputs and returns $P(Y \leq k)$.
   (c) Using only a random number generator that generates values uniformly distributed between 0 and 1, write a function that takes inputs $\lambda$ and $n$ and generates a sequence of $n$ independent random values drawn from a $\mathcal{P}\text{ois}(\lambda)$ distribution. Hint: for each draw, generate a uniform random number, $u$, between 0 and 1, and return $k$ if

   $$P(Y \leq k-1) < u \leq P(Y \leq k)$$

   . Explain why this algorithm samples from the correct distribution.
   (d) Using the function you wrote in 8c, compute the mean of a random sample of size $n$ for increasing values of $n$. As $n$ gets larger, the sample mean will get closer to $\mathbb{E}[Y]$. Use this fact to determine $\mathbb{E}[Y]$ in terms of $\lambda$. (Don't cheat and look up the Poisson distribution!).
   (e) Do the same thing to find the variance.

9. **Monte Carlo Integration.** Probability can be useful to get numerical solutions to calculus problems that may have nothing to do with probability. Consider the integral

$$\int_{-\infty}^{\infty} \cos(x)e^{-x^2}\ dx$$

You will have a hard time finding an analytic solution to the above without some fairly hefty real analysis tools; however, we can get an approximate numerical solution using random variables. Write a script that calculates this integral by representing the integral as $\mathbb{E}\left[g(X)\right]$ where $X$ is a random variable with a known distribution, and approximating this expectation with

$$\frac{1}{N}\sum_{n=1}^{N} g(X_n)$$

where $X_1, \ldots, X_N$ are sampled independently from the known distribution. You do not have to write the sampling code yourself; you may use a pre-written function to do that part. Check that your numerical answer approaches the solution given by a solver such as Wolfram Alpha.

10. **Sum of Two Random Variables.** We know how to find $\mathbb{E}\left[X+Y\right]$ and and $\mathbb{V}\text{ar}\left[X+Y\right]$, but we have not yet seen how to find the distribution of the random variable $X+Y$ itself. Let $X$ and $Y$ be arbitrary random variables, and define $Z = X+Y$. Consider what must happen for $Z$ to take a particular value, $z$. If $X$ has the value 0 (say), then $Z = z$ if and only if $Y = z - 0$. If $X = 1$ then $Z = z$ if $Y = z - 1$. And so on. In general,

$$p_{X,Z}(x,z) = p_{X,Y}(x, z-x) = p_X(x)p_{Y|X}(z-x|x)$$

Thus by marginalization,

$$p_Z(z) = \sum_x p_X(x)p_{Y|X}(z-x|x)$$

(a) Suppose $X$ and $Y$ are independent discrete uniform random variables each with range $R = \{1,2,3,4\}$. Find the formula for $P(X+Y=z)$.

(b) Now write a function to simulate draws from $X+Y$. Do this many times and calculate the proportion of the time that $X+Y=z$ for various $Z$. Does the simulated distribution match your analytic formula?

11. **Multivariate Gaussian.** (Adapted from FCMA Ex. 2.5) Assume that $p(\mathbf{w})$ is the Gaussian PDF for a $D$-dimensional vector $\mathbf{w}$ given in

$$p(\mathbf{w}) = \frac{1}{(2\pi)^{D/2}|\mathbf{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{w}-\mu)^\top \mathbf{\Sigma}^{-1}(\mathbf{w}-\mu)\right\}.$$

By expanding the vector notation and re-arranging, show that using $\mathbf{\Sigma} = \sigma^2\mathbf{I}$ as the covariance matrix assumes independence of the $D$ elements of $\mathbf{w}$. You will need to be aware that the determinant of a matrix that only has entries on the diagonal ($|\sigma^2\mathbf{I}|$) is the product of the diagonal values and that the inverse of the same matrix is constructed by simply inverting each element on the diagonal. (Hint, a product of exponentials can be expressed as an exponential of a sum. Also, just a reminder that $\exp\{x\}$ is $e^x$.)

12. **Conditional Expectation.** Show that the following holds.

$$\mathbb{E}\left[\mathbb{E}\left[Y|X\right]\right] = \mathbb{E}\left[Y\right]$$

where the inner expectation is taken with respect to the conditional distribution of $Y$ given $X$, and the outer expectation is taken with respect to the marginal distribution of $X$. You can show this for either the discrete or continuous case (up to you); the steps are the same whether you are working with sums and PMFS or integrals and PDFs.