STAT 336: HW FOR 2/9/16

(1) (Exercise 1.11) Suppose A and B are independent events (recall from the exercise on day 1 that even if we define independence asymmetrically to start, the symmetric relation of the conventional definition holds). Show that A^c and B^c are independent as well.

(2) Use induction to show that for any set of events $A_1, \ldots, A_n$,

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \prod_{i=2}^{n} \mathbb{P}(A_i \mid A_1 \cap \cdots \cap A_{i-1})$$

(3) (slightly modified from Exercise 1.18) Suppose k events, $A_1, \ldots, A_k$ form a partition of the sample space, Ω. Let B be an event satisfying $\mathbb{P}(B) > 0$. Prove that if $\mathbb{P}(A_1 \mid B) > \mathbb{P}(A_1)$, then $\mathbb{P}(A_i \mid B) < \mathbb{P}(A_i)$ for some $i = 2, \ldots, k$, and similarly if $\mathbb{P}(A_1 \mid B) < \mathbb{P}(A_1)$, then $\mathbb{P}(A_i \mid B) > \mathbb{P}(A_i)$ for some $i = 2, \ldots, k$. (That is, if we have k possible hypotheses, and some data increases (resp. decreases) the plausibility of one of them, then the plausibility of at least one other must correspondingly decrease (resp. increase).)

(4) (Exercise 1.10 – Monty Hall Problem) A prize is placed at random behind one of three doors. You pick a door (suppose you always pick door 1). Now Monty Hall (the gameshow host) chooses one of the other two doors, opens it and shows you that it is empty (he can always do this, of course, since there is always at least one of the other two doors that does not contain the prize). He then gives you the opportunity to either keep your door or switch to the other unopened door. Should you stay or switch? Intuition suggests it doesn't matter. However, the correct answer is that the probability you will win the prize is twice as high if you switch. Prove this. It will help to specify the sample space and the relevant events carefully. It is suggested that you let $\Omega = \{(\omega_1, \omega_2) : \omega_i \in \{1, 2, 3\}\}$, where ω_1 is where the prize is and ω_2 is the door Monty opens.

(5) (Exercise 1.12) There are three cards. The first is green on both sides, the second is red on both sides, and the third is green on one side and red on the other. We choose a card at random and we see one side (also chosen at random). If the side we see is green, what is the probability that the other side is also green? Many people intuitively answer 1/2. Show that the correct answer is 2/3.

(6) (Exercise 1.20a-b) A box contains 5 coins and each has a different probability of showing heads. Let $p_1, \dots, p_5$ denote the probability of heads on each coin. Suppose that

$$p_1 = 0, p_2 = 1/4, p_3 = 1/2, p_4 = 3/4, p_5 = 1$$

Let H denote "heads is obtained" and let C_i denote the event that coin i is selected.

(a) Select a coin (uniformly) at random and toss it. Suppose a head is obtained. What is the posterior probability that coin i was selected ($i = 1, \dots, 5$)? That is, find $\mathbb{P}(C_i \mid H)$ for each $i = 1, \dots, 5$.

(b) What is the probability of getting a second head from the same coin, whichever it happens to be (that is, without stipulating which coin it is)? In other words, find $\mathbb{P}(H_2 \mid H_1)$ where $H_j =$ "heads on toss j".

(7) (Exercise 2.6) Let X have CDF F and PDF f and let $A \subset \mathbb{R}$. Let $I_A(x)$ be the indicator function for A, that is:

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 x \notin A \end{cases}$$

Define $Y = I_A(X)$. Find expressions for f_Y and F_Y.

(8) (modified from Exercise 2.15) Let $U \sim \text{Uniform}(0, 1)$ and define $X^* = F^{-1}(U)$, where F is a valid CDF. Show that $X^* \sim F$. (I outlined how to show this in class; fill in the details.)

(9) Let $X_n \sim \mathsf{Binom}(n, \lambda/n)$, $n = 1, 2, \dots$. Let f_n be the PMF of X_n, and let f be the PMF of a Poisson(λ) random variable. Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ for arbitrary nonnegative integer x.